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Characteristic relations for quantum matrices

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Abstract. The general algebraic properties of the algebras of vector fields over the quantum linear groups $GL_q(N)$ and $SL_q(N)$ are studied. These quantum algebras appear to be quite similar to the classical matrix algebra. In particular, the quantum analogues of the characteristic polynomial and characteristic identity are obtained for them. The q-analogues of the Newton relations connecting two different generating sets of central elements of these algebras (the determinant-like and trace-like ones) are derived. This allows one to express the q-determinant of quantized vector fields in terms of their q-traces.

1. Introduction

Since their discovery, quantum groups have been presented in several closely related but not strictly equivalent forms. Having originally been obtained as the quantized universal enveloping (QUE) algebras [1, 2], they were then reformulated in a matrix form [3]. In this latter approach a quantum group is generated by a pair of upper- and lower-triangular matrix generators L_+ and L_- satisfying quadratic permutation relations. A further variation of this approach is to combine L_+ and L_- into a single matrix generator $L = S(L_-)L_+$. Here $S(\cdot)$ is the usual notation for the antipodal mapping. Following [4] we will call the algebra generated by the matrix generator L the reflection equation algebra (REA). After suitable treatment this algebra can be related to the quantum group by the (Hopf algebra) isomorphism [5, 6], although the Hopf structure is implicit in the REA formulation. This algebra has found several applications (see [4, 7, 8, 9] and references therein). Let us mention only one of them here, namely the construction of the quantum group differential calculus, in which the matrix generators L are used as the basic set of (right-)invariant vector fields [10–12].

A remarkable property of the REA formulation is that the algebra of the L matrices turns out to be quite similar in several respects to the classical matrix algebra. In particular, both the notions of the matrix trace and the matrix determinant admit generalization to the case of L matrices (see [3, 13] and [6, 14, 15]). In the present paper we intend to establish further similarities of the REA to the classical matrix algebra. We restrict ourselves to considering the REA of the $GL_q(N)$ and/or $SL_q(N)$ type only. For these algebras the recurrent formulae relating two different centre generating sets, the determinant-like and the trace-like ones, are obtained. These formulae are quantum analogues of the classical

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Newton relations. Furthermore, we define the characteristic polynomial and derive the characteristic identities (the analogue of the Cayley–Hamilton theorem) for the L-matrices. The existence of these identities was first mentioned in [4], where they were presented for the case $N = 2$ (see also [16]). For general N the characteristic identities were obtained in [17] (see remark 4.8 of [17]) when studying the algebraic structure of quantum Yangians $Y_q(gl(N))$. We reproduce this result in the REA approach. Then, by joint use of the quantum Newton relations and the characteristic identities one can obtain the expressions for trace-like central elements of higher powers. In the QUE representation the similar characteristic identities were considered in [18]. We believe that the REA representation and the use of the R-matrix technique makes all considerations and the final formulae much more transparent.

We conclude this section with a brief mention of some facts from the classical theory of matrices (see, e.g., [19]), which will then be generalized to the quantum case.

Consider the $N \times N$ matrix A with complex entries. Its characteristic polynomial is defined as

$$\Delta(x) \equiv \det \| x \mathbb{1} - A \| \equiv x^N + \sum_{k=1}^{N} (-1)^k \sigma(k)\, x^{N-k} . \tag{1.1}$$

The eigenvalues $\{\lambda_i\}$, $i = 1, \ldots, N$, of the matrix A are solutions of the characteristic equation $\Delta(x) = 0$. The coefficients $\sigma(i)$ of the characteristic polynomial expressed in terms of λ_i form the set of basic symmetric polynomials of N variables:

$$\begin{aligned} \sigma(1) &\equiv \sum_{i=1}^{N} \lambda_i = \operatorname{Tr} A \\ \sigma(2) &\equiv \sum_{i<j} \lambda_i \lambda_j \\ &\;\vdots \\ \sigma(N) &\equiv \prod_{i=1}^{N} \lambda_i = \det A . \end{aligned} \tag{1.2}$$

One can also express $\sigma(i)$ directly in terms of the matrix elements of A. Up to a numerical factor each $\sigma(i)$ is given by the sum of all the principal minors of the ith order

$$\sigma(i) = \frac{1}{i!\,(N-i)!}\, \epsilon^{1\ldots N} A_1 \ldots A_i\, \epsilon^{1\ldots N} . \tag{1.3}$$

Here $\epsilon^{1\ldots N}$ is the antisymmetric Levi–Civita N-tensor. The compact matrix notations used in this formula will be explained later (directly in the quantum case).

Another standard set of symmetric polynomials is given by the traces of powers of the matrix A

$$s(i) \equiv \sum_{k=1}^{N} (\lambda_k)^i = \operatorname{Tr} A^i \qquad 1 \leqslant i \leqslant N . \tag{1.4}$$

The two basic sets $\{\sigma(i)\}$ and $\{s(i)\}$ are connected by the so-called Newton relations

$$i\,\sigma(i) - s(1)\,\sigma(i-1) + \cdots + (-1)^{i-1} s(i-1)\,\sigma(1) + (-1)^i s(i) = 0 . \tag{1.5}$$

In particular, these recurrent relations allow one to express the determinant of the matrix A as a polynomial of the traces of its powers.

Finally, if one substitutes the matrix A in the characteristic polynomial (1.1) instead of the scalar variable x then the resulting matrix expression vanishes identically. This is the Cayley–Hamilton theorem, and according to it any function of the matrix A can be reduced to a polynomial of an order not exceeding $N - 1$.

2. Quantum Newton relations and the characteristic polynomial

First of all let us introduce some definitions and notation to be used in what follows. The REA is defined as the algebra generated by matrix generators L subject to the following permutation rules:

$$L_1\,\hat{R}_{12}\,L_1\,\hat{R}_{12} = \hat{R}_{12}\,L_1\,\hat{R}_{12}\,L_1\,. \tag{2.1}$$

Here the standard notation for matrix spaces (see [3]) is used: $\hat{R}_{12}$ is the $GL_q(N)$ R-matrix [2] satisfying the Yang-Baxter equation and the Hecke condition respectively

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23} \tag{2.2}$$

$$\hat{R}^2 = \mathbf{1} + \lambda\hat{R} \tag{2.3}$$

where $\mathbf{1}$ is the unit matrix, and $\lambda = q - 1/q$. Below we will further compress this notation by denoting $\hat{R}_{i(i+1)} \equiv \hat{R}_i$ and omitting the index of the L-matrix, i.e. $L_1 \equiv L$, since it always appears in the first matrix space.

We will also need the notions of the quantum trace and the q-deformed Levi–Civita tensor. The operation Tr_q of taking the quantum trace of a $N \times N$ quantum matrix X looks like

$$\mathrm{Tr}_q(X) = \mathrm{Tr}(\mathcal{D}X) \qquad \mathcal{D} = \mathrm{diag}\{q^{-N+1}, q^{-N+3}, \ldots, q^{N-1}\}\,. \tag{2.4}$$

The q-deformed Levi–Civita tensor $\epsilon_q^{i_1\ldots i_N}$ (or $\epsilon_q^{1\ldots N}$ in brief notation) is defined, up to a factor by its characteristic property

$$\left(\hat{R}_i + \frac{1}{q}\right)\epsilon_q^{1\ldots N} = 0 \qquad 1 \leqslant i \leqslant N-1\,. \tag{2.5}$$

The normalization is usually fixed by demanding $\epsilon_q^{i_1\ldots i_N} = 1$ for $i_1 = 1, \ldots, i_N = N$. Its square is then equal to

$$|\epsilon_q|^2 \equiv \epsilon_q^{1\ldots N}\epsilon_q^{1\ldots N} = q^{N(N-1)/2}N_q!,$$

where $p_q = (q^p - q^{-p})/\lambda$ are usual q-numbers. Note, that both of these definitions are closely related to the Hopf structure of the quantum group and the corresponding comodule structure of the REA [3, 20].

Two generating sets for the centre of the REA were presented in [3]. One of them is formed by the trace-like elements

$$s_q(i) = q^{1-N}\,\mathrm{Tr}_q\,L^i \qquad 1 \leqslant i \leqslant N \tag{2.6}$$

where the normalizing factor is chosen for convenience. Another generating set consists of the determinant-like elements $\sim \epsilon_q^{1\ldots N}L_{-N}\ldots L_{-(i+1)}L_{+i}, \ldots L_{+1}\epsilon_q^{1\ldots N}$. For our purposes it is better to express these generators in terms of the L-matrices:

$$\sigma_q(i) = \alpha_i\epsilon_q^{1\ldots N}(L_1\hat{R}_1\ldots\hat{R}_{i-1})^i\epsilon_q^{1\ldots N}. \tag{2.7}$$

Here the α_i are normalizing constants. The first of them is fixed as

$$\alpha_1 = q^{1-N}N_q/|\epsilon_q|^2$$

by the natural condition $\sigma_q(1) = s_q(1)$. Others will be specified below.

The connection between the two basic sets $\{\sigma_q(k)\}$ and $\{s_q(k)\}$ is provided by quantum analogues of the Newton relations (1.5). At the classical level they are usually derived by using the spectral representations (1.2), (1.4) for $\sigma(k)$ and $s(k)$. However, this representation is not available in the quantum case. Indeed, the spectrum of the quantum matrix L can be constructed only if the centre of the REA is algebraically closed. The latter in turn can be treated only in a particular representation of the REA. To overcome this difficulty we shall develop a little bit more technique.

Define an operator S_N which symmetrizes any $N \times N$ matrix X in N matrix spaces:

$$S_N(X) = X_1 + \hat{R}_1 X_1 \hat{R}_1 + \cdots + \hat{R}_{N-1} \ldots \hat{R}_1 X_1 \hat{R}_1 \ldots \hat{R}_{N-1}. \tag{2.8}$$

The characteristic properties of this symmetrizer

$$\left[S_N(X)\,,\, \hat{R}_i\right] = 0 \qquad 1 \leqslant i \leqslant N-1 \tag{2.9}$$

are fulfilled due to relations (2.2), (2.3). Furthermore, the following useful formula:

$$\epsilon_q^{1\ldots N} S_N(X) = s_X \epsilon_q^{1\ldots N} \tag{2.10}$$

is a direct consequence of (2.9) and (2.5). Here the scalar factor s_X reads

$$s_X = \frac{1}{|\epsilon_q|^2} \epsilon_q^{1\ldots N} S_N(X) \epsilon_q^{1\ldots N} = q^{1-N}\, \mathrm{Tr}_q\, X\,.$$

For $X = L^i$ this factor coincides with $s_q(i)$ (2.6). Now we are able to prove

Proposition. For $1 \leqslant i \leqslant N$ the generators $\sigma_q(i)$ and $s_q(i)$ are connected by the relations

$$\frac{i_q}{q^{i-1}} \sigma_q(i) - s_q(1)\, \sigma_q(i-1) + \cdots + (-1)^{i-1} s_q(i-1)\, \sigma_q(1) + (-1)^i s_q(i) = 0 \tag{2.11}$$

provided that the numerical factors α_i are fixed as follows:

$$\alpha_i = \frac{N_q!}{(N-i)_q!\, i_q!}\, \frac{q^{-i(N-i)}}{|\epsilon_q|^2}\,. \tag{2.12}$$

Proof. Consider the quantities $s_q(i-p)\sigma_q(p)$ for $1 \leqslant p \leqslant i-1$. With the help of (2.10) and the definitions of $s_q(i)$ and $\sigma_q(i)$ one can perform the following transformations:

$$\begin{aligned}
s_q(i-1)\sigma_q(1) &= \alpha_1 \underline{s_q(i-1)\epsilon_q^{1\ldots N}} L\, \epsilon_q^{1\ldots N} = \alpha_1 \epsilon_q^{1\ldots N} S_N(L^{i-1}) L\, \epsilon_q^{1\ldots N} \\
&= s_q(i) + \alpha_1 \frac{(N-1)_q}{q^{N-2}} \epsilon_q^{1\ldots N} L^{i-1} R_1 L\, R_1 \epsilon_q^{1\ldots N} \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots \\
s_q(i-p)\sigma_q(p) &= \alpha_p \frac{p_q}{q^{p-1}} \epsilon_q^{1\ldots N} (L^{i-p+1} R_1 \ldots R_{p-1})(L\, R_1 \ldots R_{p-1})^{p-1} \epsilon_q^{1\ldots N} \\
&\quad + \alpha_p \frac{(N-p)_q}{q^{N-p-1}} \epsilon_q^{1\ldots N} (L^{i-p} R_1 \ldots R_p)(L\, R_1 \ldots R_p)^p \epsilon_q^{1\ldots N} \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots \\
s_q(1)\sigma_q(i-1) &= \alpha_{i-1} \frac{(i-1)_q}{q^{i-2}} \epsilon_q^{1\ldots N} (L^2 R_1 \ldots R_{i-2})(L\, R_1 \ldots R_{i-2})^{i-2} \epsilon_q^{1\ldots N} \\
&\quad + \alpha_{i-1} \frac{(N-i+1)_q}{q^{N-i}}\, \frac{\sigma_q(i)}{\alpha_i}\,.
\end{aligned}$$

Now the arbitrary coefficients α_p should be fixed in such a way that the last term in $s_q(i-p+1)\sigma_q(p-1)$ and the first one in $s_q(i-p)\sigma_q(p)$ are equal. This is the case if the α_p satisfy the relations

$$\alpha_p = q^{2p-1-N} \frac{(N-p+1)_q}{p_q} \alpha_{p-1}\,. \tag{2.13}$$

Then, on taking the alternating sum $\sum_p (-1)^{p-1} s_q(i-p)\sigma_q(p)$ we find that the only terms which survive are the first one in $s_q(i-1)\sigma_q(1)$ and the last one in $s_q(1)\sigma_q(i-1)$ and, thus, we obtain relations (2.11). Finally, given the value of α_1 one easily shows that the recursion (2.13) is solved by (2.12). □

A few remarks are in order here:

(i) So far we have always been considering the matrices $GL_q(N)$. Specializing to the $SL_q(N)$ case can be achieved by fixing the quantum determinant of the L-matrix (see [6, 14, 15]): $\operatorname{Det} L = q^{1-N}\sigma_q(N) = 1$.
(ii) It is worth mentioning that the singular points where the connection between $\{s_q(i)\}$ and $\{\sigma_q(i)\}$ breaks down are the roots of unity: $k_q = 0$ for $1 \leqslant k \leqslant N$. This is apparently related to the fact that the isomorphism of the Hecke algebra of the A_{N-1} type and the group algebra of symmetric group $\mathbb{C}S_N$ is also destroyed at these points (see, e.g., [21]).

Now let us turn to the derivation of the quantum characteristic identity for the matrix L. It can be found in a way quite similar to that of the classical case. Namely, we should find a matrix polynomial B of $(N-1)$th order in L obeying the relation

$$(L - x\mathbb{1})\, B(L,x)\, \epsilon_q^{1\ldots N} = \epsilon_q^{1\ldots N}\, \Delta\,(x)\,. \tag{2.14}$$

Here x is a $\mathbb{C}$-number variable and $\Delta(x)$ is a scalar polynomial of x, the characteristic polynomial of the L-matrix. The following theorem is a generalization of the Cayley–Hamilton theorem to the quantum case.

Theorem. The matrix polynomial $B(L,x)$, when defined as

$$B(L,x) = \hat{R}_1 \ldots \hat{R}_{N-1} \prod_{i=1}^{N-1} \left[(L - q^{2i} x\mathbb{1}) \hat{R}_1 \ldots \hat{R}_{N-1} \right] \tag{2.15}$$

satisfies the relation (2.14). The characteristic polynomial of the matrix L looks like

$$\Delta\,(x) = \sum_{i=0}^{N} (-x)^i \sigma_q(N-i) \tag{2.16}$$

and for the L-matrix the following characteristic identity is satisfied:

$$\Delta(L) = \sum_{i=0}^{N} (-L)^i \sigma_q(N-i) \equiv 0\,. \tag{2.17}$$

Proof. The relation (2.14) is fulfilled if and only if its left-hand side is totally q-antisymmetric, i.e. if it obeys the characteristic relations (2.5) of the q-antisymmetric tensor. This in turn is valid if the matrix quantity $(L - x\mathbb{1})\, B(L,x)$ commutes with all $\hat{R}_i$, $1 \leqslant i \leqslant N-1$ up to terms proportional to the q-symmetric projectors $P_{+i} \equiv (\hat{R}_i + 1/q)/2_q$, which vanish when being contracted with $\epsilon_q^{1\ldots N}$. Now, the key observation is that the commutator $[\hat{R}_1\,,\,(L - x\mathbb{1})\hat{R}_1(L - \beta x\mathbb{1})\hat{R}_1]$ is proportional to P_+ only if $\beta = q^2$. With this observation the construction of B becomes clear and one immediately checks that B chosen as in (2.15) does satisfy the relation (2.14).

Then, as a direct consequence of (2.14) and (2.15) we get the following expression for $\Delta(x)$:

$$\Delta(x) = \frac{1}{|\epsilon|^2} \epsilon_q^{1\ldots N} \prod_{i=0}^{N-1} \left[(L - q^{2i} x\mathbb{1}) \hat{R}_1 \ldots \hat{R}_{N-1} \right] \epsilon_q^{1\ldots N}\,.$$

This expression can be further simplified with the use of (2.5), (2.1), (2.2) and the q-combinatorial relations. The calculations are straightforward but rather lengthy and we omit them here, presenting only the result in (2.16).

To prove the characteristic identity we shall contract the relation (2.14) with $\epsilon_q^{2\ldots N+1}$:

$$(L - x\mathbf{1})\,\epsilon_q^{2\ldots N+1} B(L,x)\,\epsilon_q^{1\ldots N} = (\epsilon_q^{2\ldots N+1}\epsilon_q^{1\ldots N})\,\Delta\,(x)\,.$$

The right-hand side of this relation is proportional to the unit matrix and, hence, $\epsilon_q^{2\ldots N+1} B \epsilon_q^{1\ldots N}$ is proportional to $(L - x\mathbf{1})^{-1}$. The classical limit of this relation is the standard base for proving the Cayley–Hamilton theorem [19]. In the quantum case all the considerations are completely the same, and the resulting statement is that the matrix polynomial $\Delta(L)$ vanishes identically. □

Here we present few final comments:

(i) The characteristic identity provides us with the compact expression for the inverse matrix of L:

$$L^{-1} = \frac{1}{\sigma_q(N)} \sum_{i=0}^{N-1} (-L)^i \sigma_q(N - i - 1)\,.$$

(ii) Multiplying the characteristic identity by L^p, and taking the q-trace we obtain the expressions of higher symmetric polynomials $s_q(N + p)$ in terms of the basic ones $\sigma_q(i)$.
(iii) On passing to concrete REA representations the order of the characteristic identity may decrease due to the basic symmetric polynomials $\sigma_q(i)$ becoming dependent. This is illustrated in the recent paper [22] where the L-matrices were realized as pseudo-differential operators acting on the quantum plane and they were found to possess the characteristic identity of second order.

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